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Symmetry analysis and similarity electrostatic waves in a nonuniform dusty magnetoplasma

Xiao-Yan Tang[1,2] **and Padma Kant Shukla**[1,3]

[1] Institut für Theoretische Physik IV, Fakultät für Physik und Astronomie, Ruhr-Universität Bochum, D-44870 Bochum, Germany
[2] Department of Physics, Shanghai Jiao Tong University, Shanghai 200240, People's Republic of China
[3] School of Physics, University of KwaZulu-Natal, 4000 Durban, South Africa

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Abstract
We apply the group theory to a (3+1)-dimensional nonlinear system relevant for the low-frequency electrostatic waves in a nonuniform dusty magnetoplasma. In correspondence with the generators of the symmetry group allowed by the system, new types of similarity reductions are performed. Some new exact solutions are obtained, which can be in the form of solitary waves, shock waves and periodic waves. Especially, our solutions indicate that the system may have time-dependent nonlinear shears. Some explicit similarity electrostatic wave solutions with time periodic nonlinear shears are displayed graphically.

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1. Introduction

During the last decade, there have been a number of theoretical and numerical studies [1, 2] of numerous waves and coherent nonlinear structures in dusty plasmas composed of the electrons, ions and charged dust grains. In different situations, different types of waves have been investigated, such as the dust acoustic and dust ion-acoustic waves [3, 4], magnetoacoustic waves [5], dust drift waves [6], coupled drift-Alfvén waves [7], dust lattice waves [8] etc. The above-mentioned waves have been studied by using the multi-fluid equations, supplemented by the Poisson or Maxwell equations, for dusty plasmas ignoring dust charge perturbation [9]. The reductive perturbation method has been frequently used to derive the Korteweg–de-Vries (KdV) equation [1], the modified KdV equation, the nonlinear Schrödinger (NLS) equation [1, 10], which admit localized structures [11, 12].

About a decade ago, Shukla *et al* [6] derived a pseudo-three-dimensional Charny–Hasegawa–Mima (C–HM) equation [1] for nonlinear electrostatic drift waves in a warm dusty magnetoplasma. Shukla and Varma [13] presented a generalized Navier–Stokes equation for two-dimensional low-frequency (in comparison with the ion gyrofrequency) convective cells

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in a nonuniform cold dusty magnetoplasma. They found that in the presence of immobile charged dust impurities, the divergence of the plasma current density associated with the $\mathbf{E} \times \mathbf{B_0}$ plasma flow remains finite, and that this contribution gives rise to a new plasma mode, which is now referred to as the Shukla–Varma (SV) mode. Both the C–HM and SV equations admit stationary solutions in the form of a dipole vortex [14, 15] and a vortex street [16].

In the presence of the parallel electron dynamics, the SV equation is then coupled with an equation that represents the acceleration of magnetic field-aligned electrons by the parallel electrostatic force. Here we will have a (3+1)-dimensional nonlinear coupled governing system of equations. In this paper, we will study the (3+1)-dimensional system to find some exact nonstationary solutions.

The manuscript is organized in the following fashion. In section 2, the derivation of our (3+1)-dimensional system governing the dynamics of low-frequency electrostatic waves in a nonuniform dusty magnetoplasma is briefly reviewed. In section 3, the classical Lie symmetries of the (3+1)-dimensional system are obtained by using the symmetry approach, and an infinite-dimensional Lie algebra of the symmetries are presented. In section 4, in correspondence to the generators, four types of (2+1)-dimensional similarity reduction equations are obtained. In section 5, some exact solutions are obtained from two of the reduced equations by using the symmetry approach again. Some explicit similarity solutions with time periodic nonlinear shears are displayed graphically. Section 6 contains the discussion and conclusions.

2. Basic equations

We consider an electron–ion–dust plasma in an external magnetic field $B_0\hat{z}$, where B_0 is the magnetic field strength, and $\hat{z}$ is the unit vector in a Cartesian coordinate system. The charged dust grains are supposed to be monosized, immobile, and inhomogeneous along the x-axis. The equilibrium charge neutrality condition reads $n_{i0}(x) = n_{e0}(x) + \delta Z_d n_{d0}(x)$, where $n_{s0}(x)$ is the unperturbed number density of the plasma species s (s equals e for the electrons, i for the ions and d for the dust grains), and Z_d is the number of charges residing on the dust grain surface. For positively (negatively) charged dust grains, $\delta = -1(+1)$.

In the presence of low-frequency (in comparison with the ion gyrofrequency $\omega_{ci} = eB_0/m_i c$, where e is the magnitude of the electron charge, m_i is the ion mass, and c is the speed of light in vacuum) electrostatic field $\mathbf{E}(= -\nabla_\perp\phi$, where ϕ is the wave potential), the electron and ion velocities in our cold dusty plasma are [1]

$$\mathbf{v}_e \approx \frac{c}{B_0}\hat{z} \times \nabla_\perp\phi + v_{ez}\hat{z}, \tag{1}$$

and

$$\mathbf{v}_i \approx \frac{c}{B_0}\hat{z} \times \nabla_\perp\phi - \frac{c}{B_0\omega_{ci}}\left(\frac{\partial}{\partial t} + \frac{c}{B_0}\hat{z} \times \nabla_\perp\phi \cdot \nabla_\perp + v_{iz}\frac{\partial}{\partial z}\right)\nabla_\perp\phi + v_{iz}\hat{z}, \tag{2}$$

where v_{ez} (v_{iz}) is the parallel (to $\hat{z}$) component of the electron (ion) fluid velocity. The electron and ion continuity equations are

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \mathbf{v}_j) = 0, \qquad j = e, i. \tag{3}$$

Inserting (1) and (2) into (3), letting $n_j = n_{j0} + n_{j1}$, where $n_{j1} \ll n_{j0}$, and assuming $(c/B_0)|\hat{z} \times \nabla_\perp\phi \cdot \nabla_\perp| \gg v_{jz}\partial/\partial z$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}n_{e1} + \frac{c}{B_0}\hat{z} \times \nabla_\perp\phi \cdot \nabla_\perp n_{e0} + n_{e0}\frac{\partial}{\partial z}v_{ez} = 0, \tag{4}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}n_{i1} + \frac{c}{B_0}\hat{z} \times \nabla_\perp \phi \cdot \nabla_\perp n_{i0} + n_{i0}\frac{\partial}{\partial z}v_{iz} - \frac{cn_{i0}}{B_0\omega_{ci}}\frac{\mathrm{d}}{\mathrm{d}t}\nabla_\perp^2 \phi = 0, \tag{5}$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + (c/B_0)\hat{z} \times \nabla_\perp \phi \cdot \nabla_\perp$.

Subtracting (4) from (5), and imposing the quasi-neutrality approximation $n_{e1} = n_{i1}$, we have

$$\frac{\partial}{\partial t}\nabla_\perp^2 \phi + \frac{c}{B_0}\left[\phi, \nabla_\perp^2 \phi\right] + \frac{\delta\omega_{ci}}{n_{i0}}\frac{\partial(Z_d n_{d0})}{\partial x}\frac{\partial \phi}{\partial y} + \frac{n_{e0}B_0\omega_{ci}}{n_{i0}c}\frac{\partial v_{ez}}{\partial z} = 0, \tag{6}$$

where the parallel component of the electron velocity perturbation v_{ez} is determined from

$$\frac{\partial v_{ez}}{\partial t} + \frac{c}{B_0}[\phi, v_{ez}] = \frac{e}{m_e}\frac{\partial \phi}{\partial z}. \tag{7}$$

The parallel component of the ion velocity perturbation v_{iz} is neglected since $v_{iz} \ll v_{ez}$. Furthermore, we have defined the Poisson bracket $[a, b] = a_x b_y - a_y b_x$.

Two comments are in order. First, equation (6) with $v_{ez} = 0$ is just the Shukla–Varma (SV) equation [13], which governs the nonlinear dynamics of finite frequency convective cell modes in a nonuniform dusty magnetoplasma. Stationary vortex solutions of the SV equation have been discussed in [13]. Second, in a uniform magnetoplasma, equations (6) and (7) are the relevant equations governing the dynamics of three-dimensional convective cells [1].

For convenience, we rewrite equations (6) and (7) as

$$\frac{\partial}{\partial t}\nabla^2 \phi + d[\phi, \nabla^2 \phi] + a\frac{\partial \phi}{\partial y} + b\frac{\partial v}{\partial z} = 0, \tag{8}$$

and

$$\frac{\partial v}{\partial t} + d[\phi, v] = \beta\frac{\partial \phi}{\partial z}, \tag{9}$$

with $v \equiv v_{ez}$, $a = \delta\omega_{ci}/n_{i0}\partial(Z_d n_{d0})/\partial x$, $b = n_{e0}B_0\omega_{ci}/n_{i0}c$, $d = c/B_0$, $\beta = e/m_e$.

3. Lie point symmetries

The Lie point symmetries of (8) and (9), having the form

$$\sigma_\phi = X\phi_x + Y\phi_y + Z\phi_z + T\phi_t - \Phi, \qquad \sigma_v = Xv_x + Yv_y + Zv_z + Tv_t - V, \tag{10}$$

with X, Y, Z, T, Φ and V being functions of the variables (x, y, z, t, ϕ, v), are the solutions of the linearized equations (8) and (9), namely,

$$\frac{\partial}{\partial t}\nabla^2 \sigma_\phi + d[\sigma_\phi, \nabla^2 \phi] + d[\phi, \nabla^2 \sigma_\phi] + a\frac{\partial \sigma_\phi}{\partial y} + b\frac{\partial \sigma_v}{\partial z} = 0, \tag{11}$$

and

$$\frac{\partial \sigma_v}{\partial t} + d[\sigma_\phi, v] + d[\phi, \sigma_v] = \beta\frac{\partial \sigma_\phi}{\partial z}, \tag{12}$$

which means that (8) and (9) are invariant under the transformations $\phi \to \phi + \epsilon\sigma_\phi$, $v \to v + \epsilon\sigma_v$, with a small parameter ϵ.

Substituting (10) into (11) and (12), eliminating the quantities ϕ_{xyy}, v_t and their higher order derivatives by means of (8) and (9), and setting zero all the coefficients of the independent terms of the polynomials of ϕ, v and their partial derivatives, we obtain an over-determined set of equations for the unknown functions X, Y, Z, T, Φ and V. Solving the determinant equations, we then obtain

$$X = -C_1 x + f, \qquad Y = -C_1 y + g, \qquad Z = C_3, \qquad T = C_1 t + C_2, \tag{13}$$

$$\Phi = -3C_1\phi + \frac{1}{d}\dot{g}x - \frac{1}{d}\dot{f}y + \frac{1}{\beta}\dot{p}z + \frac{a}{2bd\beta}\dot{f}z^2 + h, \qquad V = -2C_1 v + \frac{a}{bd}\dot{f}z + p, \tag{14}$$

where f, g, h, p are arbitrary functions of t, C_1, C_2, C_3 are arbitrary constants, and the dot over the function means its derivative with respect to time. The presence of these arbitrary functions and constants leads to an infinite-dimensional Lie algebra of symmetries. A general element of this algebra is written as

$$\underline{V} = C_1\underline{V}_1 + C_2\underline{V}_2 + C_3\underline{V}_3 + \underline{V}_4(f) + \underline{V}_5(g) + \underline{V}_6(h) + \underline{V}_7(p), \tag{15}$$

where

$$\begin{aligned}
\underline{V}_1 &= -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} - 3\phi\frac{\partial}{\partial\phi} - 2v\frac{\partial}{\partial v},\\
\underline{V}_2 &= \frac{\partial}{\partial t},\\
\underline{V}_3 &= \frac{\partial}{\partial z},\\
\underline{V}_4(f) &= f\frac{\partial}{\partial x} - \left(\frac{1}{d}\dot{f}y - \frac{a}{2bd\beta}\ddot{f}z^2\right)\frac{\partial}{\partial\phi} + \frac{a}{bd}\dot{f}z\frac{\partial}{\partial v},\\
\underline{V}_5(g) &= g\frac{\partial}{\partial y} + \frac{1}{d}\dot{g}x\frac{\partial}{\partial\phi},\\
\underline{V}_6(h) &= h\frac{\partial}{\partial\phi},\\
\underline{V}_7(p) &= \frac{1}{\beta}\dot{p}z\frac{\partial}{\partial\phi} + p\frac{\partial}{\partial v},
\end{aligned}$$

construct a basis for the vector space. The associated Lie algebra among these vector fields becomes

$$\left(\begin{array}{cccccccc}
 & \underline{V}_1 & \underline{V}_2 & \underline{V}_3 & \underline{V}_4(f) & \underline{V}_5(g) & \underline{V}_6(h) & \underline{V}_7(p)\\
\underline{V}_1 & 0 & -\underline{V}_2 & 0 & \underline{V}_4(f+t\dot{f}) & \underline{V}_5(g+t\dot{g}) & \underline{V}_6(3h+t\dot{h}) & \underline{V}_7(2p+t\dot{p})\\
\underline{V}_2 & & 0 & 0 & \underline{V}_4(\dot{f}) & \underline{V}_5(\dot{g}) & \underline{V}_6(\dot{h}) & \underline{V}_7(\dot{p})\\
\underline{V}_3 & & & 0 & \underline{V}_7\left(\frac{a}{bd}\dot{f}\right) & 0 & 0 & \underline{V}_6\left(\frac{1}{\beta}\dot{p}\right)\\
\underline{V}_4(f) & & & & 0 & \underline{V}_6\left(\frac{1}{\beta}(fg)'\right) & 0 & 0\\
\underline{V}_5(g) & & & & & 0 & 0 & 0\\
\underline{V}_6(h) & & & & & & 0 & 0\\
\underline{V}_7(p) & & & & & & & 0
\end{array}\right),$$

where the entry in the jth row and the kth column represents the commutator $[\underline{v}_j, \underline{v}_k]$, and $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$, $\{\underline{v}_1, \underline{v}_4, \underline{v}_5, \underline{v}_6, \underline{v}_7\}$, $\{\underline{v}_3, \underline{v}_6, \underline{v}_7\}$ are some of the subalgebras.

Let us now consider a point transformation $G : (x, y, z, t, \phi, v) \to (\xi, \eta, \zeta, \tau, U, W)$. By using seven vector elements in (15), we have the corresponding seven one-parameter groups of symmetries of (8) and (9) as

$$G_1 : (x, y, z, t, \phi, v) \to (xe^{-\epsilon}, ye^{-\epsilon}, te^{\epsilon}, \phi e^{-3\epsilon}, ve^{-2\epsilon}),$$

$$G_2 : (x, y, z, t, \phi, v) \to (x, y, z, t+\epsilon, \phi, v),$$

$$
\begin{aligned}
&G_3 : (x, y, z, t, \phi, v) \to (x, y, z+\epsilon, t, \phi, v),\\
&G_4 : (x, y, z, t, \phi, v) \to \left(x+\epsilon f, y, z, t, \phi - \frac{\epsilon}{d}\dot{f}y + \frac{a\epsilon}{2bd\beta}\dot{f}z^2, v + \frac{\epsilon a}{bd}\dot{f}z\right),\\
&G_5 : (x, y, z, t, \phi, v) \to \left(x, y+\epsilon g, z, t, \phi + \frac{\epsilon}{d}\dot{g}x, v\right),\\
&G_6 : (x, y, z, t, \phi, v) \to (x, y, z, t, \phi+\epsilon h, v),\\
&G_7 : (x, y, z, t, \phi, v) \to \left(x, y, z, t, \phi + \frac{\epsilon}{\beta}\dot{p}z, v+\epsilon p\right).
\end{aligned}
\tag{16}
$$

It is seen that G_1 is a scaling for all variables with different ratios, G_2, G_3, G_6 and G_7 are translations, and G_4, G_5 are Galilean boosts. Hence, if ϕ and v are solutions of (8) and (9), so are then $U(\xi, \eta, \zeta, \tau)$ and $W(\xi, \eta, \zeta, \tau)$.

4. 2+1 similarity reductions

After determining the infinitesimal generators, the similarity variables can be found by solving the characteristic equations

$$
\frac{\mathrm{d}x}{X} = \frac{\mathrm{d}y}{Y} = \frac{\mathrm{d}z}{Z} = \frac{\mathrm{d}t}{T} = \frac{\mathrm{d}\phi}{\Phi} = \frac{\mathrm{d}v}{V}. \tag{17}
$$

Specifically, four types of similarity reduction solutions are possible.

(1) The first type of reductions. For the most general generator $\underline{\mathrm{V}}$, we obtain the first type of similarity solutions

$$
\begin{aligned}
\phi = {} & \frac{U}{(C_1t+C_2)^3} + \frac{\dot{g}(C_1t+C_2) - C_1g}{d(C_1t+C_2)^2}x + \frac{C_1f - \dot{f}(C_1t+C_2)}{d(C_1t+C_2)^2}y\\
& + \left(\frac{aC_1^2 f}{bd\beta(C_1t+C_2)^3} - \frac{C_1a\dot{f}}{(C_1t+C_2)^2bd\beta} + \frac{a\ddot{f}}{2(C_1t+C_2)^2bd\beta}\right)z^2\\
& + \left(\frac{3C_1af}{bd\beta(C_1t+C_2)^3} - \frac{2C_1p}{(C_1t+C_2)^2\beta} + \frac{2C_1^2}{\beta(C_1t+C_2)^3}\int p\,\mathrm{d}t\right.\\
& \left. - \frac{2C_1^2a}{bd\beta(C_1t+C_2)^3}\int \frac{f}{C_1t+C_2}\mathrm{d}t + \frac{-a\dot{f} + \dot{p}bd(C_1t+C_2)}{(C_1t+C_2)^2ebd}\right)z\\
& + \frac{(-C_1b\beta g + a)f}{bd\beta(C_1t+C_2)^3} - \frac{p}{\beta(C_1t+C_2)^2} - \frac{C_1(2\ln(C_1t+C_2)-3)}{\beta(C_1t+C_2)^3}\int p\,\mathrm{d}t\\
& + \frac{C_1a(2\ln(C_1t+C_2)-3)}{bd\beta(C_1t+C_2)^3}\int \frac{f}{C_1t+C_2}\mathrm{d}t + \frac{2C_1}{d(C_1t+C_2)^3}\int g\dot{f}\,\mathrm{d}t\\
& - \frac{2C_1a}{bd\beta(C_1t+C_2)^3}\int \frac{f\ln(C_1t+C_2)}{C_1t+C_2}\mathrm{d}t - \frac{2C_1}{(C_1t+C_2)^3}\int h(C_1t+C_2)\,\mathrm{d}t\\
& + \frac{2C_1}{\beta(C_1t+C_2)^3}\int p\ln(C_1t+C_2)\,\mathrm{d}t + \frac{h}{C_1t+C_2},
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
v = {} & \frac{W}{(C_1t+C2)^2} + \left(\frac{-aC_1f}{(C_1t+C_2)^2bd} + \frac{a\dot{f}}{(C_1t+C_2)bd}\right)z + \frac{p}{C_1t+C_2}\\
& - \frac{1}{bd(C_1t+C_2)^2}\left(af + C_1bd\int p\,\mathrm{d}t - \int \frac{C_1af}{C_1t+C_2}\,\mathrm{d}t\right),
\end{aligned}
\tag{19}
$$

with U and W being similarity reduction functions with respect to the similarity variables

$$\xi = C_1xt + C_2x - f, \qquad \eta = C_1yt + C_2y - g, \qquad \zeta = z - \frac{1}{C_1}\ln(C_1t + C_2). \tag{20}$$

Here C_1, C_2, C_3, f, g, h and p have been redefined for simplicity (which is also valid for the following solutions). The similarity functions U and W satisfy the first type of similarity reduction equations

$$\left[\left(C_1\xi - d\frac{\partial U}{\partial\eta}\right)\frac{\partial}{\partial\xi} + \left(C_1\eta + d\frac{\partial U}{\partial\xi}\right)\frac{\partial}{\partial\eta} - \frac{\partial}{\partial\zeta} - C_1\right] \times \left(\frac{\partial^2 U}{\partial\xi^2} + \frac{\partial^2 U}{\partial\eta^2}\right) + a\frac{\partial U}{\partial\eta} + b\frac{\partial W}{\partial\zeta} = 0, \tag{21}$$

$$C_1\eta\frac{\partial W}{\partial\eta} + C_1\xi\frac{\partial W}{\partial\xi} - \frac{\partial W}{\partial\zeta} - \beta\frac{\partial U}{\partial\zeta} + d\frac{\partial U}{\partial\xi}\frac{\partial W}{\partial\eta} - d\frac{\partial U}{\partial\eta}\frac{\partial W}{\partial\xi} - 2C_1W = 0. \tag{22}$$

(2) The second type of reductions. For $C_1 = 0$, we have the similarity solution

$$\phi = \frac{\dot g}{C_2d}x - \frac{\dot f}{C_2d}y + \frac{a\ddot f}{C_2bd\beta}z^2 + \frac{bdC_2\dot p - a\dot f}{bd\beta C_2^2}z + \frac{af - bdC_2p + bd\beta C_2^2h}{bd\beta C_2^3} + U, \tag{23}$$

$$v = \frac{a\dot f}{C_2bd}z + \frac{p}{C_2} - \frac{f}{C_2^2bd} + W, \tag{24}$$

where U and W are similarity reduction functions with respect to the similarity variables

$$\xi = x - \frac{f}{C_2}, \qquad \eta = y - \frac{g}{C_2}, \qquad \zeta = z - \frac{t}{C_2}. \tag{25}$$

The second type of similarity reduction equations is

$$\left[dC_2\frac{\partial U}{\partial\xi}\frac{\partial}{\partial\eta} - dC_2\frac{\partial U}{\partial\eta}\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\zeta}\right]\left(\frac{\partial^2 U}{\partial\xi^2} + \frac{\partial^2 U}{\partial\eta^2}\right) + aC_2\frac{\partial U}{\partial\eta} + bC_2\frac{\partial W}{\partial\zeta} = 0, \tag{26}$$

$$\frac{\partial W}{\partial\zeta} + C_2d\frac{\partial W}{\partial\xi}\frac{\partial U}{\partial\eta} - C_2d\frac{\partial W}{\partial\eta}\frac{\partial U}{\partial\xi} + \beta C_2\frac{\partial U}{\partial\zeta} = 0. \tag{27}$$

(3) The third type of reductions. For $C_3 = 0$, we have the similarity solution

$$\begin{aligned}\phi = {} & \frac{U}{(C_1t + C_2)^3} + \left(\frac{\dot g}{d(C_1t + C_2)} - \frac{C_1g}{d(C_1t + C_2)^2}\right)x + \left(\frac{C_1f}{d(C_1t + C_2)^2}\right. \\ & \left. - \frac{\dot f}{d(C_1t + C_2)}\right)y + \frac{a}{bd\beta}\left(\frac{C_1^2f}{(C_1t + C_2)^3} - \frac{C_1\dot f}{(C_1t + C_2)^2} + \frac{\ddot f}{C_1t + C_2}\right)z^2 \\ & + \left(\frac{2C_1^2p}{e(C_1t + C_2)^3} - \frac{2C_1\dot p}{e(C_1t + C_2)^2} + \frac{\ddot p}{e(C_1t + C_2)}\right)z + \frac{\dot h}{C_1t + C_2} \\ & - \frac{2C_1h}{(C_1t + C_2)^2} - \frac{C_1}{d(C_1t + C_2)^3}\left(fg - 2\int(dC_1h + \dot f g)\,\mathrm{d}t\right),\end{aligned} \tag{28}$$

$$v = \frac{W}{(C_1t + C_2)^2} + \frac{a(C_1t + C_2)\dot f - C_1f}{bd(C_1t + C_2)^2}z + \frac{(C_1t + C_2)\dot p - C_1p}{(C_1t + C_2)^2}, \tag{29}$$

where U and W are similarity reduction functions with respect to the similarity variables

$$\xi = C_1xt + C_2x - f, \qquad \eta = C_1yt + C_2y - g, \qquad \zeta = z. \tag{30}$$

The corresponding similarity reduction equations are

$$\left[\left(C_1\xi - d\frac{\partial U}{\partial \eta}\right)\frac{\partial}{\partial \xi} + \left(C_1\eta + d\frac{\partial U}{\partial \xi}\right)\frac{\partial}{\partial \eta} - C_1\right]\left(\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2}\right) + a\frac{\partial U}{\partial \eta} - b\frac{\partial W}{\partial \zeta} = 0, \tag{31}$$

$$\left(C_1\xi - d\frac{\partial U}{\partial \eta}\right)\frac{\partial W}{\partial \xi} + \left(C_1\eta + d\frac{\partial U}{\partial \xi}\right)\frac{\partial W}{\partial \eta} - \beta\frac{\partial U}{\partial \zeta} - 2C_1 W = 0. \tag{32}$$

(4) The fourth type of reductions. For $C_3 = C_1 = 0$, we have the similarity solution

$$\phi = \frac{\dot{g}}{C_2 d}x - \frac{\dot{f}}{C_2 d}y + \frac{a\ddot{f}}{2bd\beta C_2}z^2 + \frac{\ddot{p}}{C_2 e}z + \frac{h}{C_2} + U, \tag{33}$$

$$v = \frac{a\dot{f}}{bdC_2}z + \frac{\dot{p}}{C_2} + W, \tag{34}$$

where the similarity reduction functions U and W are with respect to the similarity variables

$$\xi = x - \frac{f}{C_2}, \qquad \eta = y - \frac{g}{C_2}, \qquad \zeta = z, \tag{35}$$

and satisfy the similarity reduction equations

$$\left[d\frac{\partial U}{\partial \xi}\frac{\partial}{\partial \eta} - d\frac{\partial U}{\partial \eta}\frac{\partial}{\partial \xi}\right]\left(\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2}\right) + a\frac{\partial U}{\partial \eta} + b\frac{\partial W}{\partial \zeta} = 0, \tag{36}$$

$$d\frac{\partial U}{\partial \xi}\frac{\partial W}{\partial \eta} - d\frac{\partial U}{\partial \eta}\frac{\partial W}{\partial \xi} - e\frac{\partial U}{\partial \zeta} = 0. \tag{37}$$

5. Some exact similarity solutions

It is seen that the (2+1)-dimensional similarity reduction equations are still very complicated, and some exact solutions cannot be deduced easily. Therefore, we will apply here the symmetry approach to the reduction equations in order to obtain (1+1)-dimensional similarity reduction equations, which are easier to be solved. In this section, we only concentrate on the two (2+1)-dimensional reduction equations (26)–(27) and (36)–(37). The other two types of (2+1)-dimensional reduction equations (21)–(22) and (31)–(32), might be more complicated due to their variable coefficients.

5.1. Solutions from the (2+1)-dimensional reduction equations (26)–(27)

By using the Lie symmetry approach, we obtain the generator of the reduction equations (26) and (27)

$$\underline{V} = B_1\frac{\partial}{\partial \xi} + B_2\frac{\partial}{\partial \eta} + B_3\frac{\partial}{\partial U} + B_4\frac{\partial}{\partial W}, \tag{38}$$

with arbitrary constants B_1, B_2, B_3 and B_4. The related similarity solutions are

$$U = B_3\eta + P, \qquad W = B_4\eta + Q, \tag{39}$$

where B_2 has been set to 1 without loss of generality, P and Q are the similarity reduction functions with respect to the similarity variables $X = \xi - B_1\eta$ and ζ. The corresponding (1+1)-dimensional similarity reduction equations are

$$bC_2\frac{\partial Q}{\partial\zeta} - aC_2B_1\frac{\partial P}{\partial X} - dC_2B_3\left(1+B_1^2\right)\frac{\partial^3 P}{\partial X^3} - \left(1+B_1^2\right)\frac{\partial^3 P}{\partial X^2\partial\zeta} + aC_2B_3 = 0, \tag{40}$$

$$\frac{\partial Q}{\partial\zeta} - dC_2B_4\frac{\partial P}{\partial X} + dC_2B_3\frac{\partial Q}{\partial X} + \beta C_2\frac{\partial P}{\partial\zeta} = 0. \tag{41}$$

Therefore, once a solution of the reduced equations (40) and (41) is obtained, an exact similarity solution of the original systems (8) and (9) follows. However, it is still not an easy matter to obtain a general solution of (40) and (41), but some special solutions are possible. Here, we directly write down two types of solutions which have been transformed back to the variables of the original system.

(a) The first type of exact solutions. If $B_4 = \beta B_3C_2$ in (41), then the original systems (8) and (9) may possess the following exact solution

$$\begin{aligned}\phi = P_1 &+ \frac{\dot g}{dC_2}x - \frac{\dot f - B_3C_2d}{dC_2}y + \frac{a\ddot f}{2bd\beta C_2}z^2 - \frac{a\dot f - bdC_2\dot p}{bdC_2^2\beta}z\\ &- \frac{p}{C_2^2\beta} + \frac{af}{bd\beta C_2^3} + \frac{h}{C_2} - \frac{B_3 g}{C_2},\end{aligned} \tag{42}$$

$$\begin{aligned}v = -\frac{aB_1P_1}{bdB_3C_2} &+ \frac{a}{bdC_2}x + \frac{aB_1 - dbB_4C_2}{bdC_2}y + \frac{a(\dot f - dB_3C_2)}{bdC_2}z\\ &+ \frac{aB_3}{bC_2}t + \frac{p}{C_2} - \frac{2af}{bdC_2^2} + \frac{(aB_1 - bdC_2B_4)g}{bdC_2^2},\end{aligned} \tag{43}$$

where $P_1 \equiv P_1(kx - kB_1y - kdB_3C_2z + k(dB_3C_2t + B_1g - f)/C_2)$ is an arbitrary function of the indicated argument, k, B_1, B_3 and C_2 are arbitrary constants.

It is noted that the above exact solution might represent solitary waves, shock waves, etc depending on the selections of the arbitrary function P_1. In addition, the presence of arbitrary time-dependent functions f, g, h and p implies the intrusion of the time-dependent nonlinear shears and time-dependent background. Here, two sets of figures are plotted to show the effects of the nonlinear periodic shears on the electrostatic potential ϕ and the parallel electron velocity v. Figure 1 shows a possible solitary wave structure produced by choosing $P_1 \propto \mathrm{sech}^2(kx - kB_1y - kdB_3C_2z + k(dB_3C_2t + B_1g - f)/C_2)$ and the time-dependent functions $h = 0$, $f, p \propto \cos(t)$, $g \propto \sin(t)$. Figure 2 displays a possible shock-like structure produced by choosing $P_1 \propto \tanh(kx - kB_1y - kdB_3C_2z + k(dB_3C_2t + B_1g - f)/C_2)$, and the time-dependent functions are fixed similar to figure 1.

It is clear from solutions (42) and (43) that the electrostatic potential ϕ has time-dependent linear shears in the x and y directions, and a time-dependent nonlinear shear in the z direction, while the parallel electron velocity v has constant linear shears in the x and y directions and a time-dependent linear shear in the z direction. As manifested in figures 1 and 2, the electrostatic potential ϕ in the x–z plane is much more influenced by the nonlinear time-dependent shear than that in the other two planes where it shares similar structures, and the time-dependent linear shear has a small effect on the electron velocity v.

(b) The second type of exact solutions. Without the condition $B_4 = \beta B_3C_2$, we can obtain a type of travelling periodic wave solutions

$$\begin{aligned}\phi = A_1\cos\left[A_2\left(kx - kB_1y - \omega z + \frac{\omega t + kB_1g - kf}{C_2}\right) + A_3\right] &+ \frac{\dot g + kdA_4C_2}{dC_2}x\\ - \frac{kdA_4B_1C_2 + \dot f - dB_3C_2}{dC_2}y + \frac{a\ddot f}{2bd\beta C_2}z^2 &- \frac{bd\beta\omega A_4C_2^2 + a\dot f - bdC_2\dot p}{bd\beta C_2^2}z\end{aligned}$$

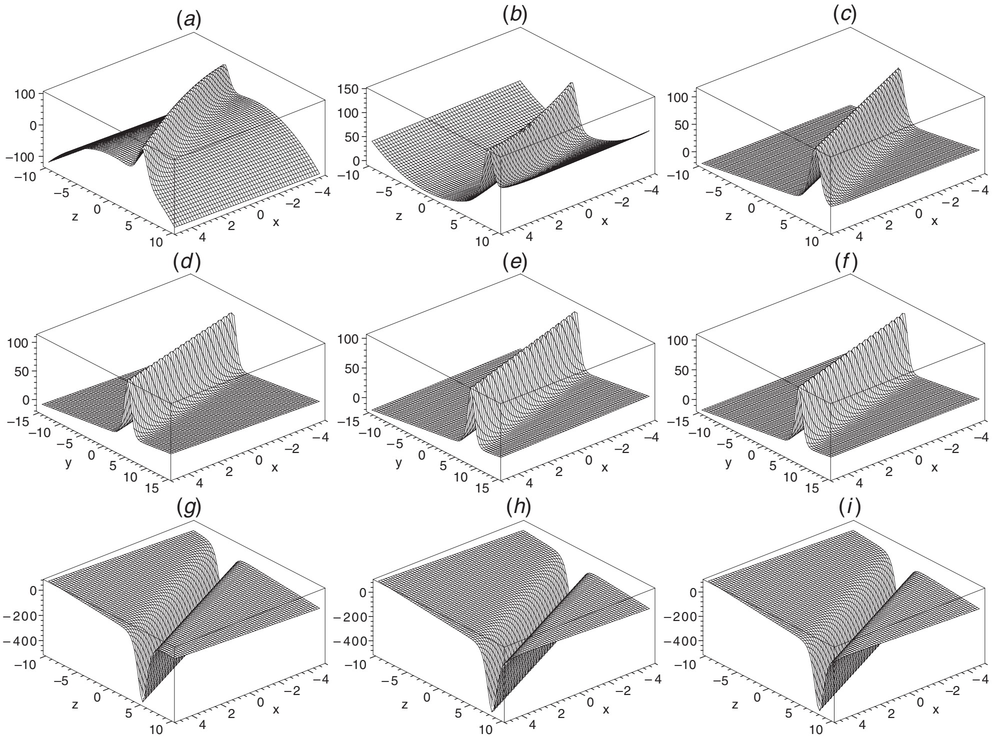

Figure 1. A schematic plot of the solitary wave with time-dependent shear at times (1) $t = 0$ for the left column, (2) $t = \frac{2}{3}\pi$ for the middle column, and (3) $t = \frac{\pi}{2}$ for the right column. The figures on the top are about the electrostatic potential ϕ on the x–z plane, in the middle are about ϕ on the x–y plane, at the bottom are about the parallel electron velocity v on the x–z plane.

$$+\frac{A_4\omega t + h + (kA_4B_1 - B_3)g}{C_2} - \frac{\left(kbd\beta A_4C_2^2 - a\right)f}{bd\beta C_2^3} - \frac{p}{\beta C_2^2}, \tag{44}$$

$$\begin{aligned} v = \frac{C_2(dkB_4 + \omega\beta)}{kdB_3C_2 - \omega}\Bigg[& A_4\left(kx - kB_1y - \omega z + \frac{\omega t + kB_1g - kf}{C_2}\right) \\ & + A_1\cos\left[A_2\left(kx - kB_1y - \omega z + \frac{\omega t + kB_1g - kf}{C_2}\right) + A_3\right]\Bigg] \\ & + B_4y + \frac{a\dot{f}}{bdC_2}z + \frac{p - B_4g}{C_2} - \frac{af}{bdC_2^2}, \end{aligned} \tag{45}$$

with

$$A_4 = \frac{aB_3(dkB_3C_2 - \omega)}{adk^2C_2B_1B_3 - akB_1\omega + bdkC_2\omega B_4 + b\omega^2\beta C_2}, \tag{46}$$

$$A_2 = \pm\sqrt{\frac{C_2(adC_2k^2B_1B_3 - akB_1\omega + bdkC_2\omega B_4 + bC_2\omega^2\beta)}{(kdB_3C_2 - \omega)^2k^2\left(B_1^2 + 1\right)}}, \tag{47}$$

where f, g, h, p are arbitrary functions of t, k, ω, B_1, B_3, C_2 and A_3 are arbitrary constants.

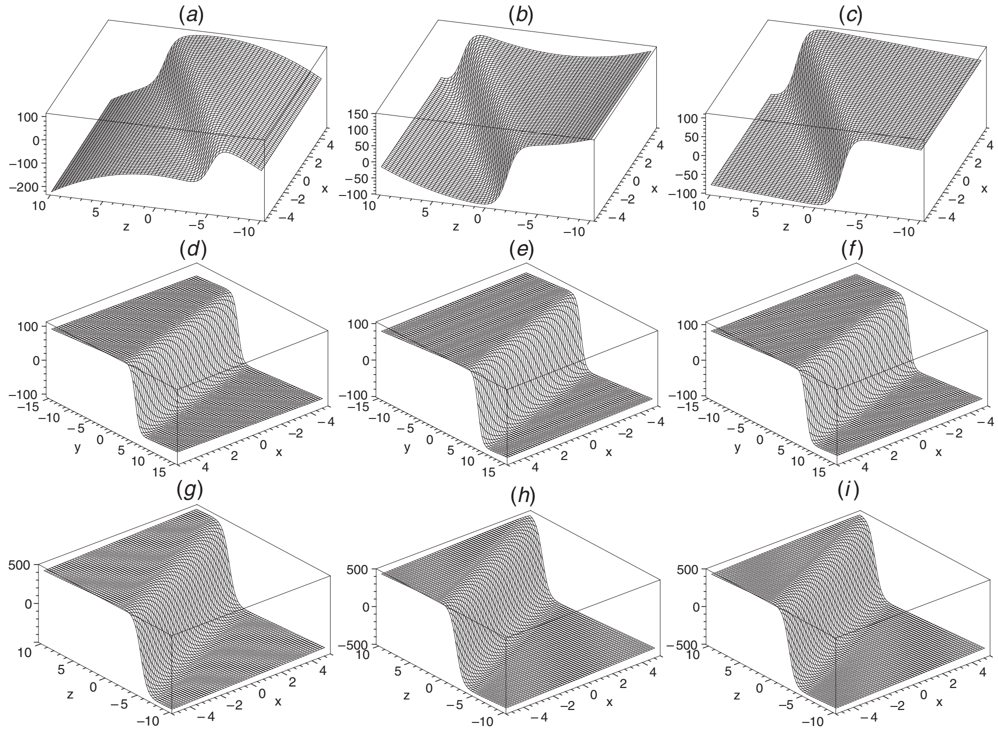

Figure 2. A schematic plot of the shock wave with time-dependent shear at times (1) $t = 0$ for the left column, (2) $t = \frac{2}{3}\pi$ for the middle column, and (3) $t = \frac{\pi}{2}$ for the right column. The figures on the top are about the electrostatic potential ϕ on the x–z plane, in the middle are about ϕ on the x–y plane, at the bottom are about the parallel electron velocity v on the x–z plane.

Figure 3 is plotted for this solution with similar selections of the time-dependent functions, namely, $h = 0$, $f, p \propto \cos(t)$, $g \propto \sin(t)$. It is observed from solutions (44)–(45) and figure 3 that, compared to the first type of solutions, the time-dependent linear and nonlinear shears appear in a similar way in this solution, and have similar effects on the electrostatic potential ϕ and the parallel electron velocity v.

5.2. Solutions from the (2+1)-dimensional reduction equations (36)–(37)

The vector field of the reduction equations (36) and (37) reads

$$\underline{\mathrm{V}} = \frac{1}{3}(B_2\xi + 3B_5)\frac{\partial}{\partial\xi} + \frac{1}{3}(B_2\eta + 3B_1)\frac{\partial}{\partial\eta} + (B_2U + B_3)\frac{\partial}{\partial U} + \frac{1}{3}(2B_2W + 3B_4)\frac{\partial}{\partial W}, \qquad (48)$$

with arbitrary constants B_1, B_2, B_3, B_4 and B_5. In this case, we can have two types of similarity solutions.

(a) The first type of exact solutions. For the most general generator (48), we can have the first type of similarity solutions

$$U = -B_3 + (\eta + 3B_1)^3 P, \qquad W = -\tfrac{3}{2}B_4 + (\eta + 3B_1)^2 Q, \qquad (49)$$

where B_2 has been set to 1 without loss of generality, P and Q are similarity reduction functions of the similarity variables $X = (\xi + 3B_5)/(\eta + 3B_1)$ and ζ, and the corresponding

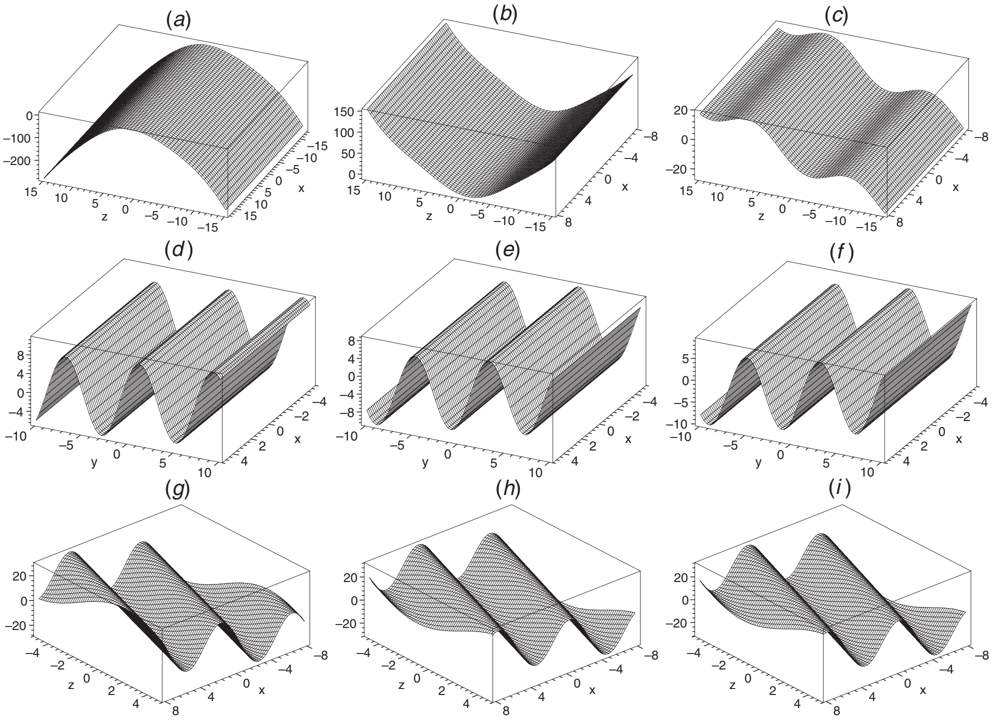

Figure 3. A schematic plot of the periodic wave with time-dependent shear at times (1) $t = 0$ for the left column, (2) $t = \frac{2}{3}\pi$ for the middle column, and (3) $t = \frac{\pi}{2}$ for the right column. The figures on the top are about the electrostatic potential ϕ on the x–z plane, in the middle are about ϕ on the x–y plane, at the bottom are about the parallel electron velocity v on the x–z plane.

(1+1)-dimensional similarity reduction equations are

$$d(1+X^2)\left(3P\frac{\partial^3 P}{\partial X^3} - \frac{\partial P}{\partial X}\frac{\partial^2 P}{\partial X^2}\right) - 6dX\frac{\partial^2 P}{\partial X^2} + 4dX\left(\frac{\partial P}{\partial X}\right)^2 + aX\frac{\partial P}{\partial X} - 3aP - b\frac{\partial Q}{\partial \zeta} = 0, \tag{50}$$

$$\beta\frac{\partial P}{\partial \zeta} + 3dP\frac{\partial Q}{\partial X} - 2dQ\frac{\partial P}{\partial X} = 0. \tag{51}$$

To obtain a general solution of the above (1+1)-dimensional similarity reduction equations is still very difficult. Below is a special polynomial type solution, which has been transformed back to the variables of the original system, read

$$\phi = \left(y^3 - \frac{3(g-3B_1C_2)}{C_2}y^2 + \frac{3(g-3B_1C_2)^2}{C_2^2}y - \frac{(g-3B_1C_2)^3}{C_2^3}\right)(A_1X^3 - 3A_2X^2 + A_3X + A_2) + \frac{\dot{g}}{C_2 d}x - \frac{\dot{f}}{C_2 d}y + \frac{a\ddot{f}}{2C_2bd\beta}z^2 + \frac{\ddot{p}}{C_2\beta}z - B_3 + \frac{h}{C_2}, \tag{52}$$

$$v = A_4\left(y^2 - \frac{2(g-3B_1C_2)}{C_2}y + \frac{(g-3B_1C_2)^2}{C_2^2}\right)(A_1X^3 - 3A_2X^2 + A_3X + A_2)^{\frac{2}{3}} + \frac{a\dot{f}}{C_2bd}z + \frac{\dot{p}}{C_2} - \frac{3}{2}B_4, \tag{53}$$

where

$$X = \frac{C_2 x - f + 3B_5 C_2}{C_2 y - g + 3B_1 C_2}, \tag{54}$$

f, g, h and p are arbitrary functions of t, $A_1, A_2, A_3, A_4, C_2, B_1, B_3$ and B_5 are arbitrary constants.

(b) The second type of exact solutions. For $B_2 = 0$, we obtain the second type of similarity solutions

$$U = \frac{B_3}{B_1}\eta + P, \qquad W = \frac{B_4}{B_1}\eta + Q, \tag{55}$$

where the similarity reduction functions P and Q, which are functions of the similarity variables $X = \xi - (B_5/B_1)\eta$ and ζ, satisfy the following (1+1)-dimensional similarity reduction equations:

$$bB_1^3 \frac{\partial Q}{\partial \zeta} - dB_3\left(B_1^2 + B_5^2\right)\frac{\partial^3 P}{\partial X^3} - aB_1^2 B_5 \frac{\partial P}{\partial X} + aB_1^2 B_3 = 0, \tag{56}$$

$$dB_4 \frac{\partial P}{\partial X} - dB_3 \frac{\partial Q}{\partial X} - eB_1 \frac{\partial P}{\partial \zeta} = 0. \tag{57}$$

By introducing a travelling wave ansatz, the above reduced equations admit a general exact solution of (8) and (9) in the form of the following periodic wave solutions:

$$\begin{aligned}\phi = A_1 \cos\left[kA_2\left(x - \frac{B_5}{B_1}y + \frac{B_5 g - B_1 f}{B_1 C_2} - \frac{\omega}{k}z\right) + A_3\right] + \frac{\dot{g} + dkA_0 C_2}{dC_2}x \\ - \frac{B_1 \dot{f} - dB_3 C_2 + kdA_0 C_2 B_5}{dC_2 B_1}y + \frac{a\ddot{f}}{2bdC_2\beta}z^2 + \frac{\ddot{p} - \omega\beta A_0 C_2}{C_2\beta}z \\ + \frac{(kA_0 B_5 - B_3)g}{B_1 C_2} - \frac{kA_0 f - h}{C_2},\end{aligned} \tag{58}$$

$$\begin{aligned}v = \frac{dkB_4 + \omega\beta B_1}{kdB_3}\left[kA_0\left(x - \frac{B_5}{B_1}y + \frac{B_5 g - B_1 f}{B_1 C_2} - \frac{\omega}{k}z\right)\right. \\ \left. + A_1 \cos\left[kA_2\left(x - \frac{B_5}{B_1}y + \frac{B_5 g - B_1 f}{B_1 C_2} - \frac{\omega}{k}z\right) + A_3\right]\right] \\ + \frac{B_4}{B_1}y + \frac{a\dot{f}}{bdC_2}z + \frac{\dot{p}}{C_2} - \frac{B_4 g}{B_1 C_2},\end{aligned} \tag{59}$$

where

$$A_0 = \frac{akdB_3^2}{b\omega^2\beta B_1^2 + bdk\omega B_4 B_1 + adk^2 B_3 B_5}, \tag{60}$$

$$A_2 = \pm\sqrt{\frac{B_1^2\left(b\omega^2\beta B_1^2 + bdk\omega B_4 B_1 + adk^2 B_3 B_5\right)}{d^2 B_3^2 k^4\left(B_1^2 + B_5^2\right)}}, \tag{61}$$

f, g, h and p are arbitrary functions of t, $k, B_1, B_3, B_4, B_5, C_2$ and ω are arbitrary constants.

It is noted that solutions (58) and (59) have forms similar to those periodic solutions (44) and (45), except with different coefficients. Hence, the time-dependent linear and nonlinear shears have a similar effect on ϕ and v, as shown in figure 3.

6. Summary and discussion

In summary, we have performed the Lie symmetry analysis of the coupled (3+1)-dimensional equations, which govern the dynamics of nonlinearly interacting low-frequency electrostatic waves in a nonuniform dusty magnetoplasma, and found their algebraic structures. In the correspondence of the generators, the system is reduced to (2+1) dimensions. Using the Lie point symmetry approach again, two of the (2+1)-dimensional systems are reduced further to some different types of (1+1)-dimensional systems. Four types of exact similarity solutions are obtained by solving the reduced (1+1)-dimensional systems.

There are three interesting features of our solutions. First, special type of solutions, given by (42) and (43), have different wave structures, such as solitary, shock, and periodic waves, depending on the arbitrary function P_1. Second, the solutions demonstrate that some types of localized waves can travel with arbitrary speed due to some arbitrary constants and time-dependent functions. Furthermore, the four exact solutions show that the electrostatic potential ϕ may not have only constant linear shears, but also time-dependent nonlinear shears. Figures 1–3 reveal that the linear and nonlinear shears have a larger effect on the electrostatic potential ϕ than on the parallel electron velocity v.

It is noted that some stationary solutions can be obtained for the (3+1)-dimensional governing system (8) and (9) by introducing a transformation of the coordinates [1]. Specifically, assuming a new coordinate $Y = y + \alpha z - ut$, where α represents the angle between the wavefront normal and the x–y plane, and u is the speed of propagation, the original system of equations are greatly simplified. From equation (9), a simple relation between the parallel electron velocity v and the electrostatic potential ϕ turns out to be

$$v = -\frac{\alpha\beta}{u}\phi. \tag{62}$$

Due to the simple relation (62), equation (8) in the stationary frame can be written in the form

$$\left[ud\frac{\partial\phi}{\partial Y}\frac{\partial}{\partial x} - \left(ud\frac{\partial\phi}{\partial x} - u^2\right)\frac{\partial}{\partial Y}\right]\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial Y^2}\right) - (ua - b\beta\alpha^2)\frac{\partial\phi}{\partial Y} = 0. \tag{63}$$

Equation (63) reduces to the Euler equation when

$$u = \frac{b\beta\alpha^2}{a}. \tag{64}$$

The Lie symmetries and similarity solutions of a (2+1)-dimensional Euler and generalized Euler equations have been studied in [17–19]. Therefore, the known solutions of the Euler and generalized Euler equations can be used to construct solutions of our equation (63) satisfying condition (64). We present two classes of analytical solutions. The first one is related to the symmetry generator

$$V = Y\frac{\partial}{\partial x} - x\frac{\partial}{\partial Y} + \left(C_1\phi - \frac{\alpha^2 b\beta}{ad}(C_1 x + Y)\right)\frac{\partial}{\partial\phi}, \tag{65}$$

and has the form

$$\begin{aligned}\phi = \frac{b\alpha^2\beta}{ad}x + \exp\left[C_1\arctan\left(\frac{Y}{x}\right)\right]\big[m_2 J\big(C_1 i, 2m_1\sqrt{x^2+Y^2}\big)\\ + m_3 N\big(C_1 i, 2m_1\sqrt{x^2+Y^2}\big)\big],\end{aligned} \tag{66}$$

where J and N are the Bessel functions of the first and second kinds, respectively, m_1, m_2, m_3 and C_1 are arbitrary constants. For nonzero C_1, (66) is a complex solution. Figure 4 shows a schematic profile of the real part of this solution in the new x–Y coordinates.

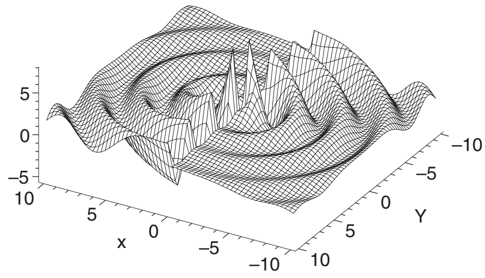

Figure 4. A schematic profile of the real part of solution (66).

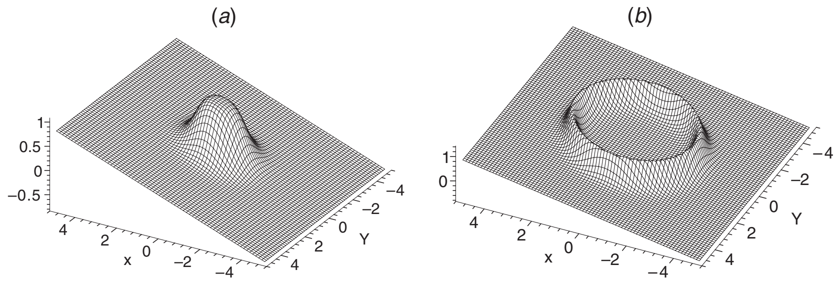

Figure 5. Schematic profiles of a dromion solution (*a*), and a ring solitary wave solution (*b*).

If $C_1 = 0$ in (65), we obtain the second interesting solution

$$\phi = \frac{b\alpha^2\beta}{ad}x + F(x^2 + Y^2), \tag{67}$$

where F is an arbitrary similarity reduction function of the indicated argument that is responsible for many interesting nonlinear coherent structures. Specifically, (67) can describe dromion solutions [20] and ring solitary waves [19, 21], when $F \propto \operatorname{sech}(x^2 + Y^2)$ and $\propto \operatorname{sech}(x^2 + Y^2 - 9)$, respectively. The dromion and ring solutions are shown in figure 5.

It is seen that solutions (66) and (67) come from the rotation transformation of the reduced system (63), which is not allowed for the original system of equations. There are two distinct features of these solutions in contrast to those presented in section 5. First, only constant linear shear in the x direction enters solutions (66) and (67). Second, due to relation (62), the electrostatic potential and the parallel electron velocity have similar structures and similar linear shears.

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